

Generalized vector field¹

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Abstract

We define generalized vector fields, and contraction and Lie derivatives with respect to them. Generalized commutators are also defined.

1 Introduction

The idea of a form of negative degree was first introduced by Sparling [1, 2]. Nurowski and Robinson [3, 4] took this idea and used it to develop a structure of generalized differential forms. A generalized p -form is an ordered pair of an ordinary p -form and a $p+1$ -form, with the wedge product of $\overset{p}{\mathbf{a}} = (\alpha_p, \alpha_{p+1})$ and $\overset{q}{\mathbf{b}} = (\beta_q, \beta_{q+1})$ being defined as $\overset{p}{\mathbf{a}} \wedge \overset{q}{\mathbf{b}} = (\alpha_p \beta_q, \alpha_p \beta_{q+1} + (-1)^q \alpha_{p+1} \beta_q)$, where α_p is an ordinary p -form, etc. The exterior derivative is defined as $\mathbf{d} \overset{p}{\mathbf{a}} = (d\alpha_p + (-1)^{p+1} k \alpha_{p+1}, d\alpha_{p+1})$. This structure was expanded to include generalized vector fields defined as an ordered pair of ordinary vector and scalar fields [5]. Here we discuss various geometric operations such as contraction, Lie derivative, commutator etc. of generalized vector fields.

2 Generalized vectors and contraction

Following [5] we define a generalized vector field as an ordered pair of an ordinary vector field v_1 and an ordinary scalar field v_0 ,

$$V := (v_1, v_0). \quad (1)$$

Clearly, the submodule $v_0 = 0$ of generalized vector fields can be identified with the module of ordinary vector fields on the manifold. Generalized scalar multiplication by a generalized zero-form $\overset{0}{\mathbf{a}} = (\alpha_0, \alpha_1)$ is defined as

$$\overset{0}{\mathbf{a}} V = (\alpha_0 v_1, \alpha_0 v_0 + i_{v_1} \alpha_1). \quad (2)$$

This is a linear operation, and satisfies $\overset{0}{\mathbf{a}} (\overset{0}{\mathbf{b}} V) = (\overset{0}{\mathbf{a}} \wedge \overset{0}{\mathbf{b}}) V$. The interior product I_V is defined as

$$I_V \overset{p}{\mathbf{a}} = (i_{v_1} \alpha_p, i_{v_1} \alpha_{p+1} + p(-1)^{p-1} v_0 \alpha_p). \quad (3)$$

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This satisfies Leibniz rule,

$$I_V(\overset{p}{\mathbf{a}} \wedge \overset{q}{\mathbf{b}}) = (I_V \overset{p}{\mathbf{a}}) \wedge \overset{q}{\mathbf{b}} + (-1)^p \overset{p}{\mathbf{a}} \wedge (I_V \overset{q}{\mathbf{b}}), \quad (4)$$

but is linear only under ordinary scalar multiplication,

$$I_{V+\mu W} = I_V + \mu I_W, \quad (5)$$

where μ is an ordinary scalar field.

3 Lie derivative

Equipped with the generalized exterior derivative and interior product we can define the Lie derivative using Cartan's formula. We will find that the resulting derivative is problematic when applied on a generalized vector field and we have to add an extra correction term. For the moment, let us define the generalized Lie derivative \mathcal{L}_V with respect to V as,

$$\mathcal{L}_V \overset{p}{\mathbf{a}} = I_V \mathbf{d} \overset{p}{\mathbf{a}} + \mathbf{d} I_V \overset{p}{\mathbf{a}}. \quad (6)$$

Since we know how to calculate the right hand side, we find

$$\begin{aligned} \mathcal{L}_V \overset{p}{\mathbf{a}} = & (L_{v_1} \alpha_p - p k v_0 \alpha_p, \quad L_{v_1} \alpha_{p+1} - (p+1) k v_0 \alpha_{p+1} \\ & + p(-1)^{p-1} (dv_0) \alpha_p + (-1)^p v_0 d\alpha_p), \end{aligned} \quad (7)$$

where as usual $\overset{p}{\mathbf{a}} = (\alpha_p, \alpha_{p+1})$, $V = (v_1, v_0)$, and L_{v_1} is the ordinary Lie derivative with respect to the ordinary vector field v_1 .

To find the definition for the Lie derivative of a vector field, we demand that the following equality holds for any two generalized vector fields V, W , and any generalized p -form $\overset{p}{\mathbf{a}}$:

$$\mathcal{L}_V(I_W \overset{p}{\mathbf{a}}) = I_W(\mathcal{L}_V \overset{p}{\mathbf{a}}) + I_{(\mathcal{L}_V W)} \overset{p}{\mathbf{a}}, \quad (8)$$

where we have written $\mathcal{L}_V W$ for the action of \mathcal{L}_V on W . This is what we would like to define as the Lie derivative of W with respect to V . Using Eqs. (3) and (7) however we find that

$$\mathcal{L}_V(I_W \overset{p}{\mathbf{a}}) - I_W \mathcal{L}_V \overset{p}{\mathbf{a}} = I_{([v_1, w_1] + k v_0 w_1, L_{v_1} w_0 - L_{w_1} v_0)} \overset{p}{\mathbf{a}} - (-1)^p (0, L_{v_0 w_1} \alpha_p), \quad (9)$$

which is not a contraction. This problem can be resolved [5] by modifying the formula for the Lie derivative of a generalized p -form to

$$\begin{aligned} \widehat{\mathcal{L}}_V \overset{p}{\mathbf{a}} &= \mathcal{L}_V \overset{p}{\mathbf{a}} + (-1)^p (0, -v_0 d\alpha_p + p dv_0 \alpha_p) \\ &= (L_{v_1} \alpha_p - p k v_0 \alpha_p, L_{v_1} \alpha_{p+1} - (p+1) k v_0 \alpha_{p+1}). \end{aligned} \quad (10)$$

This new and improved generalized Lie derivative satisfies the Leibniz rule,

$$\widehat{\mathcal{L}}_V(\overset{p}{\mathbf{a}} \wedge \overset{q}{\mathbf{b}}) = (\widehat{\mathcal{L}}_V \overset{p}{\mathbf{a}}) \wedge \overset{q}{\mathbf{b}} + \overset{p}{\mathbf{a}} \wedge (\widehat{\mathcal{L}}_V \overset{q}{\mathbf{b}}). \quad (11)$$

With this modified Lie derivative we find

$$\widehat{\mathcal{L}}_V I_W - I_W \widehat{\mathcal{L}}_V = I_{([v_1, w_1] + kv_0 w_1, L_{v_1} w_0)}. \quad (12)$$

Therefore the generalized Lie derivative of a generalized vector field is

$$\widehat{\mathcal{L}}_V W = ([v_1, w_1] + kv_0 w_1, L_{v_1} w_0). \quad (13)$$

The commutator of two generalized Lie derivatives is also a generalized Lie derivative itself,

$$\widehat{\mathcal{L}}_V \widehat{\mathcal{L}}_W - \widehat{\mathcal{L}}_W \widehat{\mathcal{L}}_V = \widehat{\mathcal{L}}_{\{V, W\}}, \quad (14)$$

which allows us define the generalized commutator as

$$\{V, W\} = ([v_1, w_1], L_{v_1} w_0 - L_{w_1} v_0). \quad (15)$$

This commutator $\{V, W\}$ is antisymmetric in V and W , bilinear and satisfies the Jacobi identity. For $U, V, W \in \mathcal{X}_G(M)$, we find that

$$\{U, \{V, W\}\} + \{V, \{W, U\}\} + \{W, \{U, V\}\} = 0. \quad (16)$$

Therefore the space $\mathcal{X}_G(M)$ of generalized vector fields together with the generalized commutator $\{, \}$ form a Lie algebra.

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